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# Reverse lex ideals

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ABSTRACT

We study reverse lex ideals in a polynomial ring, and compare their properties to those of lex ideals. In particular we provide an analogue of Green's Theorem for reverse lex ideals. We also compare the Betti numbers of strongly stable and square-free strongly stable monomial ideals to those of reverse lex ideals.


## 1. Introduction

In this paper $k$ stands for a field. We work over the polynomial ring $S = k[x_1, \dots, x_n]$ which is graded by setting the degree of each variable to be one. Throughout, $I$ stands for a monomial ideal, and we denote by $I^{\#}_j$ the set of degree $j$ monomials in $I$. We order the variables of $S$ as follows: $x_1 > \cdots > x_n$.

An *initial lex segment* of length $i$ in degree $j$ is the set of monomials consisting of the first $i$ monomials of degree $j$ in the lexicographic order. Initial lex segments have the distinction of generating as little as possible in the next degree. A monomial ideal $L$ is called *lexicographic* (or lex) if each space $L_j$ is spanned by an initial lex segment. A monomial ideal $B$ is called *strongly stable* if whenever $m$ is a minimal monomial generator of $B$, $x_i$ divides $m$, and $j < i$, we have that $x_j \cdot \frac{m}{x_i}$ is an element of $B$. Lex ideals are examples of strongly stable ideals. Both lex and strongly stable ideals play an important role in the study of Hilbert functions.

Given the importance of lex ideals, it is natural to think of defining a notion of a reverse lex ideal. In his paper [Dee96] Todd Deery considers the following version of a reverse lex ideal. He calls a monomial ideal $U$ a revlex segment ideal if $U^{\#}_j$ is an initial segment in the reverse lex order for each degree $j$. He proves [Dee96, Theorem 3.10] that such an ideal has smallest Betti numbers among all strongly stable ideals with the same Hilbert function. By [Dee96, Corollary 3.5] the Hilbert polynomial

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of a revlex segment ideal is constant, thus often there exists no revlex segment ideal attaining a given Hilbert function.

In their paper on the Betti numbers of monomial ideals [NR09], Nagel and Reiner began studying the situation in which we do not fix the Hilbert function, but only fix the number of minimal monomial generators and their degrees. Given a monomial ideal, we associate to it a reverse lex ideal (possibly in a bigger polynomial ring) as defined in Construction 1.1 below. The idea for this construction comes from [NR09].

**Construction 1.1.** Let $I \subseteq S$ be a monomial ideal and let $q_j$ be the number of minimal generators of $I$ in degree $j$ (note that $q_j$ may be 0). We construct a monomial ideal $C$ by choosing the minimal generators as follows:

For each $j \geqslant 0$, the degree $j$ minimal generators of $C$ are the $q_j$ largest monomials in the revlex order not in $\{x_1, \dots, x_n\}(C)^{\#}_{j-1}$.

It is possible for the ring $S$ not to have enough monomials in some degree in order to choose the minimal generators for $C$ in this way. An example of this is Example 2.1 and we give a way to get around this difficulty by adding extra variables.

**Definition 1.2.** Let $I$ be a monomial ideal in the ring $S$. The ideal $C$ described in 1.1 is called the *reverse lex ideal associated to* $I$.

In [NR09] Nagel and Reiner work with square-free reverse lex ideals (defined below) rather than the reverse lex ideal which we have defined.

**Definition 1.3.** The *square-free reverse lex ideal* associated to a monomial ideal $I$ is the monomial ideal $D$ constructed as in Construction 1.1 with the modification that in each degree the generators of $D$ are chosen to be the largest possible square-free monomials in the reverse lex order.

We prove in Section 4 that if $I$ is a monomial ideal then the square-free reverse lex ideal associated to $I$ and the reverse lex ideal associated to $I$ have the same Betti numbers, and hence we use the two interchangeably.

Nagel and Reiner [NR09] proposed the idea that in some cases the total Betti numbers of a square-free reverse lex ideal are smaller than or equal to the total Betti numbers of ideals with the same fixed number of minimal generators in a single degree. In general, there are examples of Hilbert functions for which no ideal has minimal Betti numbers [Ric01,DMMR07]. There are techniques for finding upper bounds on Betti numbers; obtaining lower bounds is much harder. Therefore it is interesting to consider any construction which may give lower bounds on Betti numbers. Nagel and Reiner show in [NR09] that if $I$ is a strongly stable ideal generated in one degree, then the Betti numbers of the square-free reverse lex ideal associated to $I$ are smaller than or equal to those of $I$. At the beginning of Section 3 we provide two examples showing that this property does not hold if $I$ is a strongly stable ideal generated in more than one degree. Both examples exist in a ring with four variables. In the first $\mathrm{pd}(I) < \mathrm{pd}(C)$ and in the second $I$ is a lex ideal. In view of these examples we consider in Section 3 the special case where both $I$ and $C$ have minimal generators in several degrees but in at most three variables. We prove that in this special case the Betti numbers of $C$ are indeed smaller than or equal to those of $I$.

In Section 4 we consider square-free strongly stable ideals. Nagel and Reiner showed that if $J$ is a square-free strongly stable ideal generated in one degree then the square-free reverse lex ideal associated to $J$ has smaller total Betti numbers than $J$. By passing to the strongly stable case, we are able to prove results for square-free strongly stable ideals generated in several degrees which are analogous to those proved for strongly stable ideals.

A major theorem on Hilbert functions is Green's Theorem [Gre89]. In order to formulate the theorem, we need some notation: For a monomial $m$ in $S$, we set $\max(m) = \max\{i \mid x_i \text{ divides } m\}$.

**Green's Theorem 1.4.** *(See [Gre89].) If $I \subseteq S$ is a strongly stable ideal and $L$ is the lexicographic ideal with the same Hilbert function as $I$, then for all $p$ we have*

$$\left|\left\{m \in L_j^{\#} \mid \max(m) \leqslant p\right\}\right| \leqslant \left|\left\{m \in I_j^{\#} \mid \max(m) \leqslant p\right\}\right|.$$

We prove the following theorem which is analogous to Green's Theorem above.

**Theorem 1.5.** *Let $I$ be a strongly stable ideal in $S$ and $C$ the corresponding revlex ideal. Then for all $p$ we have*

$$\left|\left\{m \in I_j^{\#} \mid \max(m) \leqslant p\right\}\right| \leqslant \left|\left\{m \in C_j^{\#} \mid \max(m) \leqslant p\right\}\right|.$$

## 2. Green's Theorem for reverse lex ideals

As stated in the introduction, the reverse lex ideal associated to a monomial ideal $I$ does not always exist in the same polynomial ring as $I$. An example of this is provided below.

**Example 2.1.** Let $S = k[a, b, c]$ and $I = (a^2, ab, ac, b^3, b^2c, bc^2, c^4)$. Then following Construction 1.1 the minimal generators for $C$ in degrees 2 and 3 are $\{a^2, ab, b^2, ac^2, bc^2, c^3\}$. There exist no monomials in degree 4 that are not divisible by these, so we cannot choose a degree 4 generator for $C$. The problem can be avoided by adding variables to the ring.

**Proposition 2.2.** *Let $I \subseteq S$ be a monomial ideal. After possibly adding variables to the ring $S$, the reverse lex ideal associated to $I$ exists. It is a strongly stable ideal.*

For the remainder of this paper we will assume the ring $S$ has sufficiently many variables to construct $C$.

For any set of monomials $M$ we define

$$W_{\leqslant p}(M) = \left\{m \in M \mid \max(m) \leqslant p\right\}$$

and

$$w_{\leqslant p}(M) = \left|\left\{m \in M \mid \max(m) \leqslant p\right\}\right|.$$

We will need the following lemma.

**Lemma 2.3.** *(See [Big93, Proposition 1.2].) If $I$ is a strongly stable ideal, then*

$$\{x_1, \ldots, x_p\} \cdot W_{\leqslant p}\left(I_j^{\#}\right) = \bigcup_{i=1}^{p} x_i \cdot W_{\leqslant i}\left(I_j^{\#}\right).$$

Now, we prove our main result:

**Theorem 1.5.** *Let $I$ be a strongly stable ideal in $S$ and $C$ the corresponding revlex ideal. Then*

$$w_{\leqslant p}\left(I_j^{\#}\right) \leqslant w_{\leqslant p}\left(C_j^{\#}\right).$$

**Proof.** We proceed by induction on $j$.

Let $\ell$ be the smallest degree in which the ideals $I$ and $C$ have minimal generators. The sets $W_{\leqslant p}(I_\ell^\#)$ and $W_{\leqslant p}(C_\ell^\#)$ consist only of minimal generators of $I$ and $C$. If $u$ and $v$ are monomials of the same degree and $\max(u) < \max(v)$, then $u > v$ in the reverse lex order. By construction, the minimal generators of $C$ in degree $\ell$ form an initial segment in the reverse lex order. So since $I$ and $C$ have the same number of minimal generators in degree $\ell$, we have the inequalities

$$w_{\leqslant p}\big(I_\ell^\#\big) \leqslant w_{\leqslant p}\big(C_\ell^\#\big)$$

for all $1 \leqslant p \leqslant n$.

Now suppose that $w_{\leqslant p}(I_{j-1}^\#) \leqslant w_{\leqslant p}(C_{j-1}^\#)$ for all $1 \leqslant p \leqslant n$. We next consider what happens in degree $j > \ell$. Fix a $p$ between 1 and $n$.

The set $W_{\leqslant p}(I_j^\#)$ consists of two kinds of monomials: minimal generators of $I$ in degree $j$ and monomials which are divisible by lower degree monomials in $I$. The latter group of monomials are exactly those in the set $\{x_1, \dots, x_p\} \cdot W_{\leqslant p}(I_{j-1}^\#)$. We know

$$\begin{aligned}
\big|\{x_1, \dots, x_p\} \cdot W_{\leqslant p}\big(I_{j-1}^\#\big)\big| &= \sum_{p}^{i=1} \big|x_i \cdot W_{\leqslant i}\big(I_{j-1}^\#\big)\big| \\
&= \sum_{p}^{i=1} w_{\leqslant i}\big(I_{j-1}^\#\big) \\
&\leqslant \sum_{i=1}^{p} w_{\leqslant i}\big(C_{j-1}^\#\big) \\
&= \sum_{p}^{i=1} \big|x_i \cdot W_{\leqslant i}\big(C_{j-1}^\#\big)\big| \\
&= \big|\{x_1, \dots, x_p\} \cdot W_{\leqslant p}\big(C_{j-1}^\#\big)\big|,
\end{aligned}$$

where Lemma 2.3 gives us the first and last equalities and the middle inequality holds by assumption. So all we need to consider are the degree $j$ minimal generators of $I$ and $C$.

By construction the degree $j$ minimal generators of $C$ were chosen to have the smallest possible maximum variables. So there are two possibilities for what happens in $C$:

*Case 1.* There are enough minimal generators in degree $j$ to exhaust the monomials in $W_{\leqslant p}(S_j^\#)$ which are not already in $\{x_1, \dots, x_p\} \cdot W_{\leqslant p}(C_{j-1}^\#)$.

In other words we have the equality,

$$w_{\leqslant p}\big(C_j^\#\big) = w_{\leqslant p}\big(S_j^\#\big).$$

This means that

$$w_{\leqslant p}\big(I_j^\#\big) \leqslant w_{\leqslant p}\big(C_j^\#\big).$$

*Case 2.* There are not enough minimal generators in degree $j$ to exhaust the monomials in $W_{\leqslant p}(S_j^\#)$.

Then all of the degree $j$ minimal generators of $C$ are in the set $W_{\leqslant p}(C_j^\#)$. Since the ideals $I$ and $C$ have the same number of degree $j$ minimal generators and since

$$\left|\{x_1, \dots, x_p\} \cdot W_{\leqslant p}\left(I^{\#}_{j-1}\right)\right| \leqslant \left|\{x_1, \dots, x_p\} \cdot W_{\leqslant p}\left(C^{\#}_{j-1}\right)\right|$$

again, we have

$$w_{\leqslant p}\left(I^{\#}_{j}\right) \leqslant w_{\leqslant p}\left(C^{\#}_{j}\right). \qquad \square$$

The theorem and the previous lemma together imply the following proposition.

**Proposition 2.4.** *An initial reverse lex segment $X$ in degree $j$ generates as much as possible in degree $j+1$ among all sets of monomials in degree $j$ with the strongly stable property and with the same cardinality as $X$.*

## 3. Betti numbers

Nagel and Reiner showed [NR09] that if $I$ is a strongly stable ideal generated in one degree and $D$ the square-free reverse lex ideal associated to $I$, then $b^S_p(D) \leqslant b^S_p(I)$ for all $p$. We construct two examples which show this is not true if $I$ is a strongly stable ideal generated in more than one degree.

**Example 3.1.** In the ring $A = k[a, b, c, d]$, let

$$I = \left(a^2, ab, ac, b^3, b^2c, bc^2, c^3\right).$$

The corresponding revlex ideal is

$$C = \left(a^2, ab, b^2, ac^2, bc^2, c^3, acd\right).$$

The Betti numbers of $I$ and $C$ are

$$\begin{array}{llll} b^A_0(I) = 7, & b^A_1(I) = 10, & b^A_2(I) = 4, & \\ b^A_0(C) = 7, & b^A_1(C) = 11, & b^A_2(C) = 6, & b^A_3(C) = 1. \end{array}$$

This example also shows that the reverse lex ideal associated to a strongly stable ideal can have higher projective dimension than the original ideal.

**Example 3.2.** Let $A = k[a, b, c, d]$ and

$$I = \left(a^2, ab, ac, ad^2, b^3, b^2c, b^2d, bc^2, bcd, bd^2, c^3\right).$$

The corresponding revlex ideal is

$$C = \left(a^2, ab, b^2, ac^2, bc^2, c^3, acd, bcd, c^2d, ad^2, bd^2\right).$$

The Betti numbers of $I$ and $C$ are

$$\begin{array}{llll} b^A_0(I) = 11, & b^A_1(I) = 22, & b^A_2(I) = 16, & b^A_3(I) = 4, \\ b^A_0(C) = 11, & b^A_1(C) = 23, & b^A_2(C) = 18, & b^A_3(C) = 5. \end{array}$$

Note that in this example, the ideal $I$ is a lexicographic ideal.

**Proposition 3.3.** *Let $I \subseteq S$ be a strongly stable ideal and $C$ the reverse lex ideal associated to $I$. If* $\max(m) \leqslant 3$ *for all the minimal generators $m$ of $I$ and $C$ then the following inequality holds for all $p$*

$$b_p^S(C) \leqslant b_p^S(I).$$

**Proof.** Let $u_1, \dots, u_r$ be the minimal generators of $I$ and $v_1, \dots, v_r$ the minimal generators of $C$. We may assume that these generators are ordered so that $\max(u_i) \leqslant \max(u_j)$ and $\max(v_i) \leqslant \max(v_j)$ for all $i < j$.

Our goal will be to use the formula for the Betti numbers of a strongly stable ideal given by the Eliahou–Kervaire resolution [EK90] to show the desired inequalities on the Betti numbers of $I$ and $C$. The Eliahou–Kervaire resolution gives the following formula for Betti numbers of a strongly stable ideal $I$

$$b_p^S(I) = \sum_{i=1}^{r} \binom{\max(u_i) - 1}{p}.$$

Therefore, it will be sufficient to show that for all $1 \leqslant i \leqslant r$

$$\max(v_i) \leqslant \max(u_i). \qquad (**)$$

Since the ideals $I$ and $C$ are strongly stable, $\max(u_1) = 1$ and $\max(v_1) = 1$, and these are the only minimal generators in either ideal which have this property. This together with the assumption that $\max(u_i) \leqslant 3$ and $\max(v_i) \leqslant 3$ for all $1 \leqslant i \leqslant r$ means that all we need to show to prove $(**)$ is

$$\bigl|\bigl\{u_i \bigm| 1 \leqslant i \leqslant r,\ \max(u_i) \leqslant 2\bigr\}\bigr| \leqslant \bigl|\bigl\{v_i \bigm| 1 \leqslant i \leqslant r,\ \max(v_i) \leqslant 2\bigr\}\bigr|.$$

Let $\ell$ be the smallest and $d$ the largest degree of a minimal generator of $I$. Then

$$\begin{aligned}
\bigl|\bigl\{u_i \bigm| 1 \leqslant i \leqslant r,\ \max(u_i) \leqslant 2\bigr\}\bigr| &= w_{\leqslant 2}\bigl(I_\ell^\#\bigr) + \sum_{j=\ell+1}^{d} \bigl(w_{\leqslant 2}\bigl(I_j^\#\bigr) - \bigl|\{x_1, x_2\} \cdot W_{\leqslant 2}\bigl(I_{j-1}^\#\bigr)\bigr|\bigr) \\
&= w_{\leqslant 2}\bigl(I_\ell^\#\bigr) + \sum_{\ell+1}^{d}\bigl(w_{\leqslant 2}\bigl(I_j^\#\bigr) - w_{\leqslant 2}\bigl(I_{j-1}^\#\bigr) - 1\bigr) \\
&= \sum_{\ell}^{d} w_{\leqslant 2}\bigl(I_j^\#\bigr) - \sum_{\ell+1}^{d} w_{\leqslant 2}\bigl(I_{j-1}^\#\bigr) - \bigl(d - (\ell + 1)\bigr) \\
&= w_{\leqslant 2}\bigl(I_d^\#\bigr) - d + \ell + 1.
\end{aligned}$$

The second equality above follows from Lemma 2.3. A similar formula holds for $C$, so by Theorem 1.5 we have the desired inequality. □

## 4. Square-free strongly stable ideals

We will find it useful to be able to pass from a square-free strongly stable ideal to the case of a strongly stable ideal, which we have already considered. To this end we recall a bijection which was introduced by Aramova, Herzog, and Hibi [AHH00] between monomials and square-free monomials in $k[x_1, x_2, \dots]$.

We think of a degree $j$ monomial (in any number of variables) as a $j$-tuple of positive integers that correspond to the subscripts of the variables. In other words, the monomial $x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_j}$ is associated to $(\alpha_1, \alpha_2, \ldots, \alpha_j)$ where the $\alpha_i$ are not necessarily distinct. When representing a monomial this way we will always assume that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_j$. We use this notation to define a function from the set of monomials to the set of square-free monomials as follows

$$\varphi : \{\text{monomials}\} \to \{\text{square-free monomials}\},$$
$$\varphi\big((\alpha_1, \ldots, \alpha_j)\big) = (\alpha_1, \alpha_2 + 1, \ldots, \alpha_i + i - 1, \ldots, \alpha_j + j - 1).$$

This function can be used to obtain square-free strongly stable ideals from strongly stable ideals and vice versa as the next proposition demonstrates. The following two propositions were proved in [AHH00].

**Proposition 4.1.** *Let $I = (u_1, \ldots, u_t)$ and $J = (v_1, \ldots, v_t)$ where $v_i = \varphi(u_i)$. Then*

(a) *$I$ is strongly stable if and only if $J$ is square-free strongly stable.*
(b) *If $I$ is strongly stable (and hence $J$ is square-free strongly stable), then $u_1, \ldots, u_t$ are the minimal generators of $I$ if and only if $v_1, \ldots, v_t$ are the minimal generators of $J$.*

**Proposition 4.2.** *Let $I$ be a strongly stable ideal with minimal generators $u_1, \ldots, u_t$ and $J = (\varphi(u_1), \ldots, \varphi(u_t))$. Then for all $p$*

$$b_p^S(I) = b_p^S(J).$$

We list two examples that illustrate the fact that if $J$ is a square-free strongly stable ideal generated in more than one degree and $D$ the square-free reverse lex ideal associated to $J$, then it is not necessarily true that the Betti numbers of $D$ are smaller than or equal to those of $J$.

**Example 4.3.** Let $A = k[a, \ldots, f]$. We will apply the function $\varphi$ to the ideals in Examples 3.1 and 3.2. Proposition 4.2 tells us that the Betti numbers of these ideals are the same as the Betti numbers of the ideals in Examples 3.1 and 3.2.

From Example 3.1 we get the square-free strongly stable ideal

$$J = (ab, ac, ad, bcd, bce, bde, cde),$$

and the square-free reverse lex ideal associated to $J$

$$D = (ab, ac, bc, ade, bde, cde, adf).$$

From Example 3.2 we get

$$J = (ab, ac, ad, aef, bcd, bce, bcf, bde, bdf, bef, cde),$$

and the square-free reverse lex ideal

$$D = (ab, ac, bc, ade, bde, cde, adf, bdf, cdf, aef, bef).$$

**Theorem 4.4.** *Let $I$ be any monomial ideal and let $C$ be the reverse lex ideal associated to $I$ and $D$ the square-free reverse lex ideal associated to $I$. Then $C$ and $D$ have the same Betti numbers.*

**Proof.** By the previous proposition, it will be sufficient to show that if $u_1, \dots, u_t$ are the minimal generators of $C$, then $D = (\varphi(u_1), \dots, \varphi(u_t))$. This is easily checked. For completeness we include the argument. We assume that $u_1, \dots, u_t$ are ordered so that $\deg(u_i) \leqslant \deg(u_{i+1})$ and if $\deg(u_i) = \deg(u_{i+1})$, then $u_i \succ_{rlex} u_{i+1}$. It is well known that $\varphi$ preserves the reverse lex order (see [AHH00]), so the same order applies to $\varphi(u_1), \dots, \varphi(u_t)$. In other words, $\deg(\varphi(u_i)) \leqslant \deg(\varphi(u_{i+1}))$ and if $\deg(\varphi(u_i)) = \deg(\varphi(u_{i+1}))$, then $\varphi(u_i) \succ_{rlex} \varphi(u_{i+1})$.

$(\varphi(u_1), \dots, \varphi(u_t))$ has the right number of minimal generators in each degree so the only possible problem is if there were some $s$ such that $\deg(\varphi(u_s)) = j$ and some square-free degree $j$ monomial $m$ such that $m \succ_{rlex} \varphi(u_s)$ and $m \notin (\varphi(u_1), \dots, \varphi(u_{s-1}))$. Then $\varphi^{-1}(m) \succ_{rlex} u_s$ which implies by the construction of $C$ that $\varphi^{-1}(m) \in (u_1, \dots, u_{s-1})$. Since $C = (u_1, \dots, u_t)$ is strongly stable and by the way $u_1, \dots, u_t$ are ordered, $(u_1, \dots, u_{s-1})$ is strongly stable also, so $\varphi^{-1}(m) = u_r w$ for some monomial $w$ and some $1 \leqslant r \leqslant s-1$ and such that $\max(u_r) \leqslant \min(w)$. Thus $m = \varphi(u_r w) = \varphi(u_r)w'$ which is a contradition. Therefore for any $1 \leqslant s \leqslant t$ if $m \succ_{rlex} \varphi(u_s)$ and $\deg(m) = \deg(\varphi(u_s))$ then $m \in (\varphi(u_1), \dots, \varphi(u_{s-1}))$. This is the defining property of $D$ so $D = (\varphi(u_1), \dots, \varphi(u_t))$ and hence $D$ and $C$ have the same Betti numbers. □

**Corollary 4.5.** *Let $J$ be a square-free strongly stable ideal and $D$ the square-free reverse lex ideal associated to $J$. If $\max(m) - \deg(m) \leqslant 2$ for all minimal generators $m$ of both $J$ and $D$ then*

$$b_p^S(D) \leqslant b_p^S(J)$$

*for all $p$.*

**Proof.** Let $I = \varphi^{-1}(J)$ and $C = \varphi^{-1}(D)$. Then the assumption $\max(m) - \deg(m) \leqslant 2$ for all minimal generators of $J$ and $D$ means that the generators of $I$ and $C$ involve at most 3 variables. Since $C$ is the reverse lex ideal associated to $I$, the claim follows by Proposition 3.3. □

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